Journal of Mathematical Analysis and Applications **231,** 235–240 (1999)
Article ID jmaa.1998.6259, available online at http://www.idealibrary.com on IDEAL®

Oscillation Criteria for a Forced Second-Order Linear Differential Equation

James S. W. Wong

Chinney Investments Limited, City University of Hong Kong, Room 814, Swire House

Received July 1, 1998

1

We are here concerned with the second-order forced linear differential equation,

$$(p(t)y')' + q(t)y = f(t), \qquad t \in [0, \infty), \tag{1}$$

where p, q, f are continuous functions, $p > 0$ and for simplicity is assumed to be continuously differentiable on $[0, \infty)$. Our interest is to establish oscillation criteria for Eq. (1) that do not assume that q and f be of definite sign. It is well known that the linear inhomogeneous Eq. (1) has unique solutions for each set of initial conditions and that such solutions are continuable throughout $[0, \infty)$. A solution is said to be *oscillatory* if it has arbitrarily large zeros; i.e., for any $T > 0$ there exist a $t \geq T$ such that $x(t) = 0$. Equation (1) is said to be oscillators if every solution is oscillatory.

In other articles on this problem, it is usually assumed that $q(t)$ be nonnegative; see Keener [5], Rainkin [7], Skidmore and Leighton [9], Skidmore and Bowers [8], and Teufel [10]. In this case, one can usually establish oscillation criteria for a more general nonlinear equation by employing a technique introduced by Kartsatos [4] where it is additionally assumed that f be the second derivative of an oscillatory function h. This approach has been explored in our paper [11]. In this paper, we do not impose such a restriction on $f(t)$, and we proceed to give two oscillation criteria for Eq. (1) which do not assume that $q(t)$ is nonnegative. Examples are given to show how these theorems can be applied where previous results are inconclusive.

0022-247X/99 $30.00

2

In this section, we extend a result of El-Sayed [3] on the oscillation of Eq. (1), namely,

THEOREM A. *Let there exist two positive increasing divergent sequences* $\{a_n^+\}, \{a_n^-\}$ *and two sequences* $\{c_n^+\}$ *and* $\{c_n^-\}$ *such that* c_n^+, c_n^- *are positive numbers and*

$$V_n^{\pm} = \int_{a_n^{\pm}}^{a_n^{\pm} + \pi/\sqrt{c_n^{\pm}}} \Big(c_n^{\pm}[1 - p(t)] \cos^2\Big\{\sqrt{c_n^{\pm}}\,(t - a_n^{\pm})\Big\} + \big[q(t) - c_n^{\pm}\big] \sin^2\Big\{\sqrt{c_n^{\pm}}\,(t - a_n^{\pm})\Big\}\Big)\, dt \geq 0, \quad (2)$$

for all $n \geq n_0$, *where* n_0 *is a fixed positive integer. Suppose that* $f(t)$ *satisfies*

$$f(t) \begin{cases} \geq 0, & t \in \left[a_n^+, a_n^+ + \dfrac{\pi}{\sqrt{c_n^+}}\right], \\[2ex] \leq 0, & t \in \left[a_n^-, a_n^- + \dfrac{\pi}{\sqrt{c_n^-}}\right], \end{cases} \quad (3)$$

for all $n \geq n_0$. *Then Eq.* (1) *is oscillatory.*

Theorem A was proved with the aid of a comparison theorem of Leighton [6] in the form given by Coppel [2; Theorem 8, p. 11]. We prove a more general result and we also provide a simpler and direct proof.

THEOREM 1. *Suppose that for any* $T \geq 0$, *there exist* $T \leq s_1 < t_1 \leq s_2 < t_2$ *such that*

$$f(t) \begin{cases} \leq 0, & t \in [s_1, t_1], \\ \geq 0, & t \in [s_2, t_2]. \end{cases} \quad (4)$$

Denote $D(s_i, t_i) = \{u \in C^1[s_i, t_i]\colon\ u(t) \not\equiv 0,\ u(s_i) = u(t_i) = 0\}$, $i = 1, 2$. *If there exists* $u \in D(s_i, t_i)$ *such that*

$$Q_i(u) = \int_{s_i}^{t_i} \big(qu^2 - pu'^2 \big) \geq 0, \quad (5)$$

for $i = 1, 2$, *then Eq.* (1) *is oscillatory.*

Proof. Suppose that $y(t)$ is a nonoscillatory solution which is eventually positive, say $y(t) > 0$ when $t \geq t_0$ for some t_0 depending on the solution $y(t)$. Denote $v(t) = -(p(t)y')/y$ for $t \geq t_0$. It follows from Eq. (1) that $v(t)$ satisfies the first-order nonlinear Riccati equation,

$$v' = \frac{v^2}{p} + q - \frac{f}{y}. \quad (6)$$

By assumption, we can choose $s_1, t_1 \geq t_0$ so that $f(t) \geq 0$ on the interval $I = [s_1, t_1]$ with $s_1 < t_1$. On the interval I, $v(t)$ satisfies by (6) the differential inequality,

$$v' \geq \frac{v^2}{p} + q, \quad \text{on } I. \tag{7}$$

Let $u(t) \in D(s_1, t_1)$ be given as in the hypothesis. Multiplying u^2 through (7) and integrating over I, we find

$$\int_I u^2 v' \geq \int_I \frac{u^2 v^2}{p} + \int_I q u^2. \tag{8}$$

Integrating (8) by parts and using the fact that $u(s_1) = u(t_1) = 0$, we obtain

$$-\int_I 2uu'v \geq \int_I \frac{u^2 v^2}{p} + \int_I q u^2,$$

which is equivalent to

$$0 \geq \int_I \left(\sqrt{p}\, u' + \frac{uv}{\sqrt{p}} \right)^2 + Q_1(u). \tag{9}$$

Because $Q_1(u) \geq 0$, (9) yields $u' - (uy'/y) = y(d/dt)(u/y) = 0$ on I. Also $y(t) > 0$, so it follows that $u(t) = Cy(t)$ for some constant C. Because $u \in D(s_1, t_1)$ and $u \not\equiv 0$, this is incompatible to the fact that $y(t) > 0$ on I. This contradiction proves that $y(t)$ must be oscillatory.

When $y(t)$ is eventually negative, we use $u \in D(s_2, t_2)$ and $f(t) \geq 0$ on $[s_2, t_2]$ to reach a similar contradiction. The proof is complete. ∎

It is easy to see that Theorem A follows from Theorem 1 by choosing $\{a_n\}, \{c_n\}$ for sufficiently large n. We now give an example to which Theorem A does not apply but Theorem 1 can be used to prove oscillation of all solutions.

EXAMPLE 1. Consider the following special case of Eq. (1),

$$\left(\sqrt{t}\, y'\right)' + y = \sin\sqrt{t}\,. \tag{10}$$

Here the zeros of the forcing term $\sin\sqrt{t}$ are $(n\pi)^2$.

Let $u = \sin\sqrt{t}$. For any $T \geq 0$, choose n sufficiently large so that $(n\pi)^2 \geq T$ and set $s_1 = (n\pi)^2$ and $t_1 = (n+1)^2\pi^2$ in (5). It is easy to verify that

$$Q_1(u) = \int_{(n\pi)^2}^{(n+1)^2\pi^2} \left[\sin\sqrt{t} - \frac{1}{4\sqrt{t}} \cos^2\sqrt{t} \right] dt = \frac{3\pi}{4}.$$

Similarly, for $s_2 = (n+1)^2\pi^2$ and $t_2 = (n+2)^2\pi^2$, we can show that $Q_2(u) > 0$. It follows from Theorem 1 that Eq. (10) is oscillatory. On the other hand, because the zeros of the forcing term grow quadratically, it is not possible to choose divergent sequences $\{a_n^+\}, \{a_n^-\}$ with an appropriate sequence of constants $\{c_n^+\}, \{c_n^-\}$ so that conditions (2) and (3) are satisfied.

3

In this section, we prove another oscillation theorem for Eq. (1) under the assumption that the unforced equation,

$$\mathscr{L}z = (p(t)z')' + q(t)z = 0 \tag{11}$$

is nonoscillatory. Let $z(t)$ be a nonprincipal solution of (11), i.e., $z(t)$ satisfies

$$\int^{\infty} \frac{dt}{pz^2} < \infty. \tag{12}$$

Define the following function $H(t)$,

$$H(t) = \int^t \frac{1}{pz^2} \left(\int^s fz \right) ds. \tag{13}$$

THEOREM 2. *Suppose that* (11) *is nonoscillatory and let* $z(t)$ *be a nonprincipal solution. Then Eq.* (1) *is oscillatory if*

$$\varlimsup_{t\to\infty} H(t) = -\varliminf_{t\to\infty} H(t) = +\infty. \tag{14}$$

Proof. The change of variable $y = z(t)w$ transforms (1) into

$$(pz^2w')' + (z\mathscr{L}z)w = fz. \tag{15}$$

When z is a solution of (11), we can express $w(t)$ by integration of (15) as follows,

$$w(t) = c_1 + c_2 \int_{t_0}^t \frac{1}{pz^2} + \int_{t_0}^t \frac{1}{pz^2} \left(\int^s fz \right) ds, \tag{16}$$

where c_1 and c_2 are constants depending on the initial conditions $w(t_0)$ and $w'(t_0)$. Note that $z(t)$ is a nonprincipal solution, so (12) and (14) imply that $w(t)$ satisfies

$$\overline{\lim_{t\to\infty}}\, w(t) = -\underline{\lim_{t\to\infty}}\, w(t) + \infty. \tag{17}$$

Because $z(t)$ is nonoscillatory so $z(t) > 0$ when $t \geq t_1$ for some large t_1. Condition (17) implies that $w(t)$ is oscillatory. Hence $y = z(t)w$ is also oscillatory. ∎

EXAMPLE 2. Consider the forced equation,

$$y'' + \frac{\sin \alpha t}{t} y = t^{\delta} \sin \beta t, \tag{18}$$

where α, β, δ are real constants. For $|\alpha| > \sqrt{2}$ the linear equation $z'' + (t^{-1} \sin \alpha t)z = 0$ is nonoscillatory. It has two linearly independent solutions z_1, z_2 having the following asymptotic behavior,

$$\begin{aligned} z_1(t) &\sim t^{\gamma+1/2} = t^{\gamma+1/2}\{1 + o(1)\}, \\ z_2(t) &\sim t^{-\gamma+1/2} = t^{-\gamma+1/2}\{1 + o(1)\}, \end{aligned} \tag{19}$$

where

$$\gamma = \left\{\frac{1}{4} - \frac{1}{2\alpha^2}\right\}^{1/2} > 0.$$

See Cassell [1, p. 292]. Here $z_1(t)$ is a nonprincipal solution. Upon substituting z_1 and $t^{\delta} \sin \beta t$ into (13), we find that (14) is satisfied if $\delta > \gamma + \frac{1}{2}$, and for any $\beta \neq 0$. Because $\gamma < \frac{1}{2}$, this shows that Eq. (18) is oscillatory for any $\delta > 1$ and for all $\alpha, \beta \neq 0$.

REFERENCES

1. J. S. Cassell, The asymptotic behaviour of a class of linear oscillators, *Quart. J. Math. Oxford Ser.* (2) **32** (1981), 287–302.
2. W. A. Coppel, "Stability and Asymptotic Behaviour of Differential Equations," Heath, Boston, 1965.
3. M. A. El-Sayed, An oscillation criterion for a forced second order linear differential equation, *Proc. Amer. math. Soc.* **118** (1993), 813–817.
4. A. G. Kartsatos, Maintenance of oscillations under the effect of a periodic forcing term, *Proc. Amer. Math. Soc.* **33** (1972), 377–383.
5. M. S. Keener, Solutions of a certain linear nonhomogeneous second order differential equations, *Appl. Anal.* **1** (1971), 57–63.

6. W. Leighton, Comparison theorems for linear differential equations of second order, *Proc. Amer. Math. Soc.* **13** (1962), 603–610.
7. S. M. Rainkin, Oscillation theorems for second order nonhomogeneous linear differential equations, *J. Math. Anal. Appl.* **53** (1976), 550–553.
8. A. Skidmore and J. J. Bowers, Oscillatory behaviour of solutions of $y'' + p(x)y = f(x)$, *J. Math. Anal. Appl.* **49** (1975), 317–323.
9. A Skidmore and W. Leighton, On the differential equation $y'' + p(x)y = f(x)$, *J. Math. Anal. Appl.* **43** (1973), 46–55.
10. H. Teufel, Jr., Forced second order nonlinear oscillations, *J. Math. Anal. Appl.* **40** (1972), 148–152.
11. J. S. W. Wong, Second order nonlinear forced oscillations, *SIAM J. Math. Anal.* **19** (1988), 667–675.